Lecture 5

5.1) Boundary Conditions.
5.2) Interface between two lossless linear media.
5.3) Interface between a dielectric and perfect conductor.
5.4) Poynting's Theorem. Poynting Vector.

The information about a word an Interface.

1) An interface is a surface forming a common boundary between two things (two liquids, materials, substances).
2) To interface = to interact= to communicate with someone, especially in a work-related situations.
3) Interface is a connection between two pieces of electronic equipment, or between a person and a computer.
4) A situation or place where things come together and affect each other.

$$\vec{J}+\frac{\partial \vec{D}}{\partial t} \Rightarrow \sigma\vec{E}+\varepsilon_0\varepsilon_r\frac{\partial \vec{E}}{\partial t};$$

$\vec{J}+\dfrac{\partial \vec{D}}{\partial t}$ is a total current.

$\partial\vec{E}$ is a conduction current.

$\varepsilon_0\varepsilon_r\dfrac{\partial \vec{E}}{\partial t}$ is a displacement current.

$\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}$ >> 1 is a conductor medium.

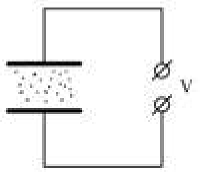
$\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}$ << 1 is a dielectric medium.

1) Perfect dielectric $\sigma=0$
2) Imperfect dielectric $\sigma\neq 0$, $\dfrac{\sigma}{\omega\varepsilon} << 1$
3) Good conductor $\dfrac{\sigma}{\omega\varepsilon} >> 1$
4) Perfect conductor $\sigma\to\infty$,

where δ is electrical conductivity. The electrical conductivity is a measure of a materials ability to conduct an electric current $\vec{I}=\sigma\vec{E}=\dfrac{\vec{E}}{\rho}$, ρ is the electrical resistivity (a measure of how strong a material opposes to flow of electric charge.

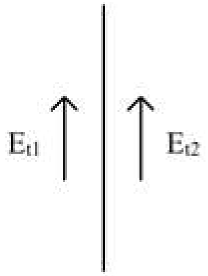
The displacement current is a quantity related to a changing electric field. It occurs in dielectric materials and also in free space

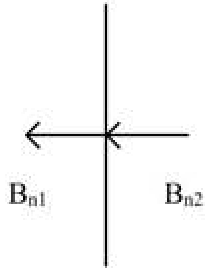
$$\varepsilon_0\varepsilon_r\frac{\partial \vec{E}}{\partial t} \longrightarrow$$



Boundary conditions
Poynting's Theorem

t – tangential components, n – normal.





$$E_{t1}-E_{t2}=0 \qquad D_{n1}-D_{n2}=q_s$$
$$B_{n1}-B_{n2}=0 \qquad H_{t1}-H_{t2}=j_s$$

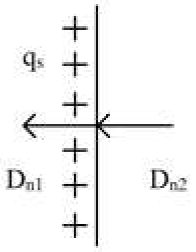
Lossy Media
Low-Loss dielectrics
Good conductors
σ [s/m] – conductivity

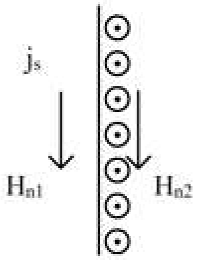
Perfect dielectric $\sigma=0$

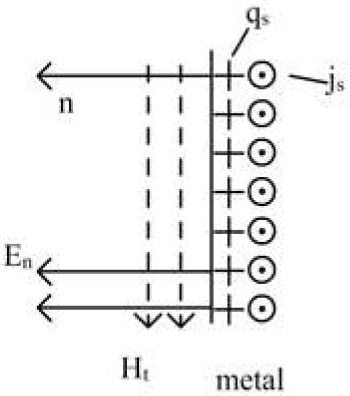
Imperfect dielectric $\sigma\neq 0$, $\frac{\sigma}{\omega\varepsilon}<<1$

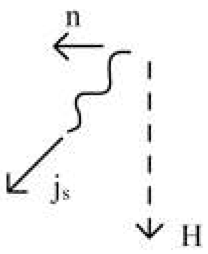
Good conductor $\frac{\sigma}{\omega\varepsilon}>>1$

Perfect conductor $\sigma\to\infty$ **(infinity)**









$E_t=0 \quad D_n=q_s$
$B_n=0 \quad H_t=j_s$

$j_s=H_t$
$q_s=D_n$

$$\vec{j}_s=[\vec{n}\vec{H}_t]$$

In order to solve electromagnetic problems involving contiguous (next to or touching another thing. The states are contiguous with each other but the laws are quite different) regions of different constitutive parameters, it is necessary to know the boundary conditions that the field vectors $\vec{E}$, $\vec{D}$, $\vec{H}$ and $\vec{B}$ must satisfy at the interfaces.
Boundary conditions are derived by applying the integral form of Maxwell's equations to a small region of an interface of two media.

1) $E_{1t} = E_{2t} \quad (V/m)$
2) $a_{n2} \times (H_1 - H_2) = J_s \quad (A/m)$
3) $a_{n2} \times (D_1 - D_2) = q_s \quad (C/m^2)$
4) $B_{1n} = B_{2n} \quad (T)$

For the time-varying case and static fields is the same.

Electromagnetic boundary conditions

1) The tangential component of the $\vec{E}$ field is continuous across an interface.
2) The tangential component of $\vec{H}$ field is discontinuous across an interface where a surface current exists, the amount of discontinuity being determined by (2).
3) The normal component of a $\vec{D}$ field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being determined by (3).
4) The normal component of a $\vec{B}$ field is continuous across an interface.

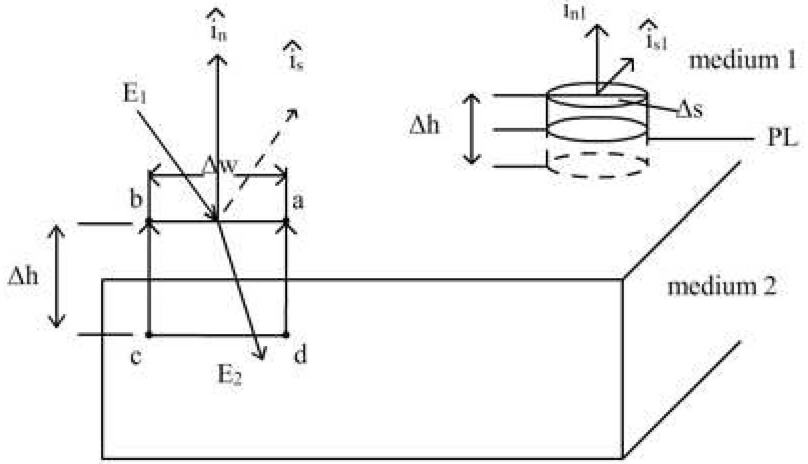
Maxwell's Equations in Large-Scale Form

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{s} \quad \text{(Faraday's law)}$$

$$\oint_L \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s} + \frac{\partial}{\partial t}\int_s \vec{D} \cdot d\vec{s} \quad \text{(Ampere's law)}$$

$$\oint_s \vec{D} \cdot d\vec{s} = \int_V \rho dv \quad \text{(Gauss's law)}$$

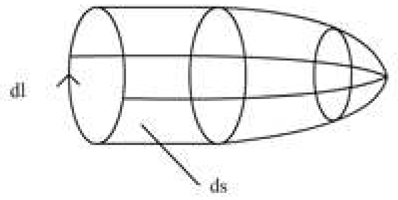
$$\oint_s \vec{B} \cdot d\vec{s} = 0$$



PL ⊥ boundary

Stokes's theorem

$$\oint_L \vec{F} \cdot d\vec{l} = \int_s (\nabla \times \vec{F}) d\vec{s}$$



The divergence theorem

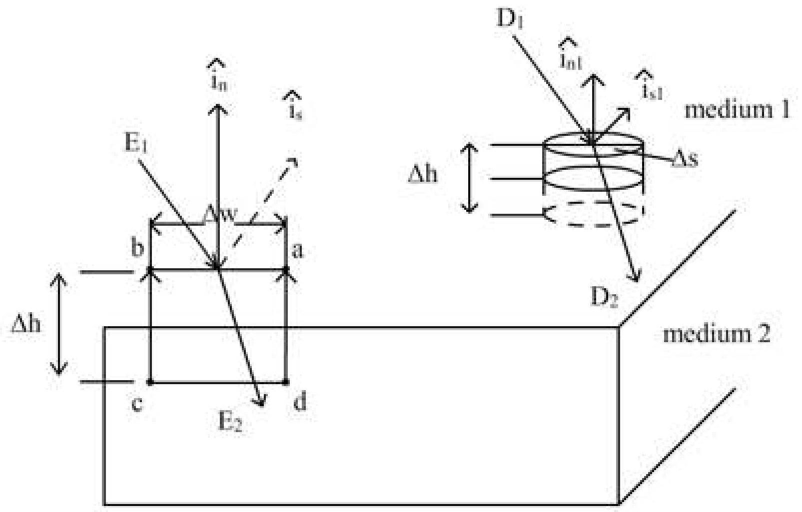
$$\int_v div\vec{F} dv = \oint_s \vec{F} d\vec{s}$$

Boundary Conditions for Electrostatic Fields

Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between two media.

For instance, we may wish to determine how the $\vec{E}$ and $\vec{D}$ vectors change in crossing an interface.

$\vec{D}$ is el. flux density.



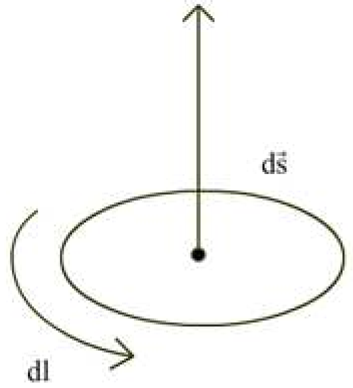
Interface between two media.
The infinitive small :
Tablet – a small solid piece of medicine = a shallow pillbox

For deriving the boundary conditions resulting from Faraday's law and Ampere's circular law.

Boundary Conditions for $E_{tangential}$

First, we consider a rectangular closed path abcd of infinite small area in the plane normal to the boundary and with it's sides "ab" and "cd" parallel to and on either side of the boundary.

Faraday's law



$$\oint_L \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$$

Applying differencing:

$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ - vector lenght

$d\vec{s} = dxdy\cdot\hat{k}$ - differential surface vector

ad and bc → and ab&cd remaining on either side of the boundary, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_L \vec{E}\cdot d\vec{l} = -\lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt}\int_s \vec{B}\cdot d\vec{s} \quad (*)$$

Where:

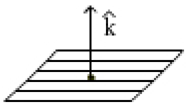
$L = ab + bc + cd + da$;

$s = area(abcd) = \Delta h\cdot\Delta w \cong 0$, where $\Delta h = 0$;

In this limit, the contributions from “ad” and “bc” to the integral on the left side of (*) approach zero. Since “ab” and “cd” are infinite small, the sum of the contributions from “ab” and “cd” becomes:

$$E_{ab}(\breve{a}b) + E_{cd}(\breve{c}d);$$
$$E_{ab} - E_{dc} = 0;$$
$$\breve{c}d = -\breve{dc};$$

In vector form: $\hat{i}_n \times (\vec{E}_1 - \vec{E}_2)$;
In scalar form: $E_{t1} - E_{t2} = 0$;



$E_{tg1} = E_{tg2} \rightarrow [v/m]$, $E_{t1} = E_{t2}$

Boundary conditions for H$_{\text{tangential}}$
(Amperes circular law, p .204 Reitz)
Applying Ampere’s circular law

$$\oint_L \vec{H}\cdot d\vec{l} = \int_s \vec{I}\cdot d\vec{s} + \frac{d}{dt}\int_s \vec{D}\cdot d\vec{s},$$

to closed path abcd in the limit that ad→0 and bc→0, we have

$$\lim_{\substack{ad\to 0\\ bc\to 0}} \oint_{abcda} \vec{H}\cdot d\vec{l} = \lim_{\substack{ad\to 0\\ bc\to 0}} \int_{\substack{area\\ abcd}} \vec{I}\cdot d\vec{s} + \lim_{\substack{ad\to 0\\ bc\to 0}} \frac{d}{dt} \int_{\substack{area\\ abcd}} \vec{D}\cdot d\vec{s}.$$

$$[H_{ab}(ab) + H_{cd}(cd)]\cdot \vec{H}_{t1}(\vec{I}_s \cdot \hat{i}_s)_{(ab)}$$
$$H_{ab} - H_{dc} = \vec{I}_s \cdot \hat{i}_s$$
$$\hat{i}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{I}_s$$

In scalar form:

$$H_{t1} - H_{t2} = I_s \text{ [A/m]}$$

The component of $\vec{H}_1$, $\vec{H}_2$ are tangential to the boundary.

Statement: At any point on the boundary, the components of $\vec{H}_1$ and $\vec{H}_2$ tangential to the boundary are discontinuous on by the amount equal to the surface current density at that point.

I→ electric current density, conduction current convection (grad T), drift → the Lorence force, $\vec{H}$.

$d\vec{l}$ is a differential length vector.

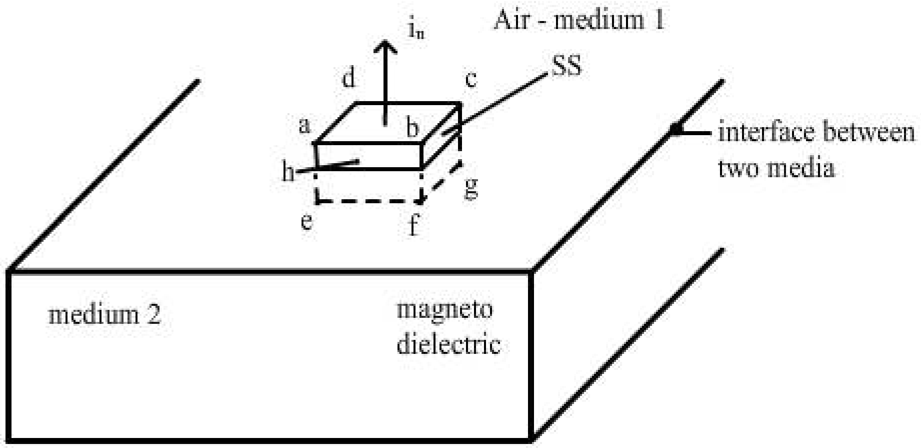
$$d\vec{l} = \hat{i}dx + \hat{j}dy + \hat{k}dz \;;$$

The differential surface vector

$$d\vec{s} = dxdy\cdot\hat{k};$$

Boundary Conditions for D_{normal}

D is an electric flux density (an electric displacement, electric induction).



Now, we consider a rectangular box abcdefg of infinite small volume.
Applying Gauss's law for the electric field:

$$\oint_s D \cdot d\vec{s} = \int_v \rho dv;$$

In the limit that the side surface (SS) tend to zero by making the volume of the box tend to zero.

$$\lim_{SS\to 0} \oint_{\substack{surface\\ of\\ the\\ box}} \vec{D} \cdot d\vec{s} = \lim_{SS\to 0} \int_{\substack{volume\\ of\\ the\\ box}} \rho \cdot dv$$

$$D_{n1}(abcd) - D_{n2}(efgh) = \rho_s(abcd)$$

$$D_{n1} - D_{n2} = \rho_s \to (c/m^2)$$

At any point on the boundary, the components of $\vec{D}_1$ and $\vec{D}_2$ normal to the boundary are discontinuous by the amount of the surface charge density ot the point.

Boundary Conditions for B_{normal}

Applying Gauss's law for the magnetic field

$$\oint_s \vec{B} \cdot d\vec{s} = 0;$$

to the box "abcdefgh" in the limit that the side surfaces field to zero, we have (ss- side surfaces):

$$\lim_{SS\to 0} \oint_{\substack{surface\\ of\\ the\\ box}} \vec{B} \cdot d\vec{s} = 0;$$

but the sides "abcd" and "efgh" remaining in either side of the boundary, we have:

$$B_{n1}(abcd) - B_{n2}(efgh) = 0;$$

Since "abcd" and "efgh" are equal (abcd = efgh):

$$B_{n1} - B_{n2} = 0$$

In terms of vectors $\vec{B}_1$ and $\vec{B}_2$

$$\hat{i}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

At any point on the boundary, the components of $\vec{B}_1$ and $\vec{B}_2$ normal to the boundary are equal.

B→[T]

Summarizing the boundary conditions, we have :

1) $\hat{i}_n \times (\vec{E}_1 - \vec{E}_2) = 0$
2) $\hat{i}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{I}_s$
3) $\hat{i}_n (D_{n1} - D_{n2}) = \rho_s$
4) $\hat{i}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0$

In scalar form:

1) $\vec{E}_1 - \vec{E}_2 = 0$
2) $\vec{H}_1 - \vec{H}_2 = \vec{I}_s$
3) $D_{n1} - D_{n2} = \rho_s$
4) $\vec{B}_1 - \vec{B}_2 = 0$

The same result would be if we consider any arbitrary- shaped boundary.
The boundary conditions given by (**) and (***) are general.

Tangential components
Normal components

1)The normal component of a $\vec{B}$ field is continuous across an interface.
2)The tangential components of an E field is continuous across interface
3)The tangential component of H field is discontinuous across an interface where a surface current exists, the amount of discontinuity being determined by $a_{n2} \times (H_1 - H_2) = J_s (A/m)$
4) The normal component of a $\vec{D}$ field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being determined by $a_{n2} \times (D_1 - D_2) = q_s (c/m^2)$

Interface between two lossless linear media

A lossless linear medium can be specified by a permittivity ε and a permeability μ, with $\delta = 0$. There are usually free charges (ρ_s) and no surface currents $\vec{I}_s$ at the interface between two lossless media.
When $\rho_s = 0$ and $\vec{I}_s = 0$ we obtain the boundary conditions listed:

1) $E_{t1} = E_{t2} \rightarrow \dfrac{D_{t1}}{D_{t2}} = \dfrac{\varepsilon_1}{\varepsilon_2}$
2) $H_{t1} = H_{t2} \rightarrow \dfrac{B_{t1}}{B_{t2}} = \dfrac{\mu_1}{\mu_2}$
3) $D_{n1} = D_{n2} \rightarrow \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$
4) $B_{n1} = B_{n2} \rightarrow \mu_1 H_{n1} = \mu_2 H_{n2}$

The index "t" means the tangential projection of the EM field. The index "n" means the normal projection.

Boundary conditions between a Dielectric (medium 1) and a Perfect Conductor (medium 2)

Medium 1 (Dielectric)	Medium 2 (metal)
$\vec{E}_{t1} = 0$	$\vec{E}_{t2} = 0$
$\hat{n} \times \vec{H} = \vec{I}_s$	$\vec{H}_{t2} = 0$
$D_{n1} = \hat{n} \cdot \vec{D}_1 = \rho_s$	$D_{n2} = 0$
$B_{n1} = \hat{n} \cdot \vec{B}_{n1} = 0$	$B_{n2} = 0$

S = Siemens

Good conductor $\frac{\sigma}{\omega \varepsilon} >> 1$

$\sigma = |q| n \mu \quad [S/m]$

q = magnitude of the current carrying particle.;
n = concentration of current;
μ = the electron and hole mobility;

Perfect dielectric ($\sigma = 0$) :

$$\hat{n} \times \vec{E} = 0$$

$$\hat{n} \times \vec{H} = \vec{I}_s$$

$$\hat{n} \cdot \vec{D} = \rho_s$$

$$\hat{n} \cdot \vec{B} = 0$$

Flow of Electromagnetic Power and the Poynting Vector

Electromagnetic waves carry with electromagnetic power. Energy is transported through space to distant receiving points by electromagnetic waves. We will now derive a relation between the rate of such energy transfer and the electric and magnetic field intensities associated with a traveling electromagnetic wave.
We begun with the curl equations :

$$\nabla \times E = -\frac{\partial B}{\partial t}; (8-77)$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}; (8-78)$$

The verification of the following identity of vector operations is straightforward :

$$\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H). \qquad (8\text{-}79)$$

Substitution of Eqs.(8-77) and (8-78) in Eq. (8-79) yields:

$$\nabla \cdot (E \times H) = -H \cdot \frac{\partial B}{\partial t} - E \cdot \frac{\partial D}{\partial t} - E \cdot J. \qquad (8\text{-}80)$$

In a simple medium, whose constitutive parameters ε, μ and σ do not change with time, we have:

$$H\cdot\frac{\partial B}{\partial t}=H\cdot\frac{\partial(\mu H)}{\partial t}=\frac{1}{2}\frac{\partial(\mu H\cdot H)}{\partial t}=\frac{\partial}{\partial t}(\frac{1}{2}\mu H^2).$$

$$E\cdot\frac{\partial D}{\partial t}=E\cdot\frac{\partial(\varepsilon E)}{\partial t}=\frac{1}{2}\frac{\partial(\varepsilon E\cdot E)}{\partial t}=\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2).$$

$$E\cdot J=E\cdot(\sigma E)=\sigma E^2.$$

Equation (8-80) can then be written as :

$$\nabla\cdot(E\times H)=-\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2)-\sigma E^2; \qquad (8-81)$$

which is a point-function relationship. An integral form of Eq. (8-81) is obtained by integrating both sides over volume of concern:

$$\oint_s(E\times H)\cdot ds=-\frac{\partial}{\partial t}\int_v(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2)dv-\int_v\sigma E^2dv; \qquad (8-82)$$

Where the divergence theorem has been applied to convert the volume integral of $\nabla\cdot(E\times H)$ to the closed surface integral of $(E\times H)$.

We recognize that the first and second terms on the right side of Eq. (8-82) represent the time-rate of change of energy stored in the electric and magnetic fields, respectively. The last term is the ohmic power dissipated in the volume as a result of the flow of conduction current density σE in the presence of the electric field E. Hence we may interpret the right side of Eq. (8-82) *as the rate of decrease* of the electric and magnetic energies stored, subtracted by the ohmic power dissipated as heat in the volume *V*. To be consistent with the law of conservation of energy, this must equal the power (rate of energy) *leaving* the volume through its surface. Thus the quantity $(E\times H)$ is a vector representing the power flow per unit area. Define:

$$\mathscr{P}=\mathrm{E}\times\mathrm{H} \qquad (\mathrm{W/m^2}) \qquad (8-83)$$

Quantity $\mathscr{P}$ is known as the **Poynting Vector**, which is a power density vector associated with an electromagnetic field. The assertion that the surface integer of $\mathscr{P}$over a closed surface, as given by the left side of Eq. (8-82), equals power leaving the enclosed volume is referred to as **Poynting theorem**. This assertion is not limited to plane waves.

Equation (8-82) may be written in another form:

$$-\oint_s P\cdot ds=\frac{\partial}{\partial t}\int_v(w_e+w_m)dv+\int_v\rho_\sigma dv, \qquad (8-84)$$

Where

$$w_e=\frac{1}{2}\varepsilon E^2=\frac{1}{2}\varepsilon E\cdot E^*=\text{the electric energy density}, \qquad (8-85)$$

$$w_m=\frac{1}{2}\mu H^2=\frac{1}{2}\mu H\cdot H^*=\text{the magnetic energy density}, \qquad (8-87)$$

$$\rho_\sigma=\sigma E^2=J^2/\sigma=\sigma E\cdot E^*=J\cdot J^*/\sigma=\text{Ohmic power density}. \quad (8-87)$$

In words, Eq.(8-84) states that the total power flowing into a closed surface at any instant equals the sum of rates of increase of the stored electric and magnetic energies and the ohmic power dissipated within the enclosed volume.

Two points concerning the Poynting vector are worthy of note. First, the power relations given in Eq. (8-82) and (8-84) pertain to the total power flow across a closed surface obtained by the surface integral of $(E\times H)$. The definition of the Poynting vector in Eq.(8-83) as the power density vector at every point on the surface is an arbitrary, albeit useful, concept. Second, the Poynting vector $\mathscr{P}$ is in a direction normal to both **E** and **H.**

If the region of concern is lossless $(\sigma = 0)$, then the last term in Eq. (8-84) vanishes, and the total power flowing into a closed surface is equal to the rate of increase of stored electric and magnetic energies in the enclosed volume. In a static situation, the first two terms on the right side of Eq. (8-84) vanish, and the total power flowing into a closed surface is equal to the ohmic power dissipated in the enclosed volume.

EXAMPLE 8-7 Find the Poynting vector on the surface of a long, straight conducting wire (of radius b and conductivity σ) that carries a direct current I. Verify Poynting's theorem.
Solution. Since we have a d-c situation, the current in the wire is uniformly distributed over its cross-sectional area. Let us assume that the axis of the wire coincides with the *z*-axis. Figure 8-8 shows a segment of length l of the long wire. We have

$$J = a_z \frac{I}{\pi b^2}$$

and

$$E = \frac{J}{\sigma} = a_z \frac{I}{\sigma \pi b^2}.$$

On the surface of the wire,

$$H = a_\phi \frac{I}{2\pi b}.$$

The Poynting vector

$\vec{P} = \vec{E} \times \vec{H}$, [W/m^2]

The vector obtained in the direction of a right-hand screw from the cross-product of the electric field vector rotated into the magnetic field vector of on electromagnetic wave

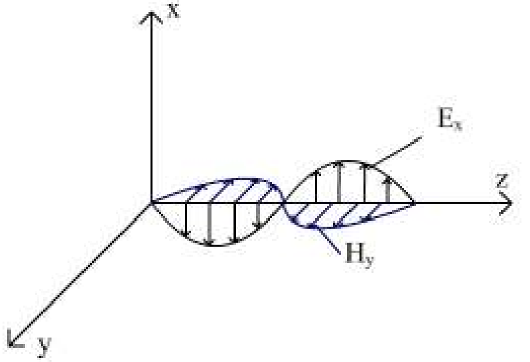
Poynting vector describes the flow-energy (power) through a surface in terms of electric and magnetic properties. The Poynting vector points in the direction of propagation of a traveling EM wave. Poynting vector has the dimensions of power per area.

History

Poynting vector is named after John Henry Poynting (1852-1914) who introduced the concept in 1884.
The Poynting vector P is defined as

power flow per unit area $\rightarrow \vec{P} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}), [W/m^2]$

μ is the permeability of the medium through witch the radiation passes,
E the electric field strength (intensity),
B is the magnetic flux density (magnetic induction).



$$\vec{E} = E_x(zmt) \cdot \hat{i}_x$$
$$\vec{H} = H_y(zmt) \cdot \hat{i}_y$$

Energy stored in the EM field is:

$$U_{EM} = \frac{1}{2}\int(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)dt;$$

Suppose we have some charge and current which, at time t, produces field $\vec{E}$ and $\vec{B}$. In the next instant, at the charges move around a bit. **Question**, how much work, *dw*, is done be the EM forces acting on those charges in the interval *dt*?

According to the Lorentz force law, the work done on a charge is:

$$\vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot vdt$$

Poynting's theorem says that the work done on the charges by the EM force is equal to the decrease in energy stored in the field, less the energy that flows out through the surface.

The energy per unit time, per unit area, transported by the fields is called the Poynting vector:

$$\vec{P} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E};$$

$$\vec{B} = \mu_0 \mu_r \vec{H};$$